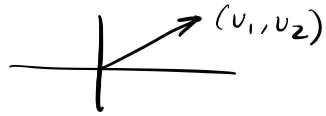
3.1 – Vectors in 2-Space, 3-Space, and n-Space

Definition: If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(v_1, v_2, \dots, v_n)$. The set of all ordered n-tuples is called **real n-space**, denoted by R^n.

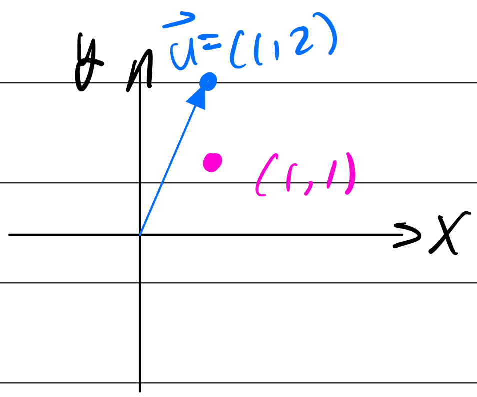


not complex

If a vector **v** in 2-space or 3-space is positioned with its initial point at the origin of a rectangular coordinate system, then the coordinates of its terminal point are **components** of **v** relative to the coordinate system. The numbers in an n-tuple can be considered coordinates of a point or components of a vector.

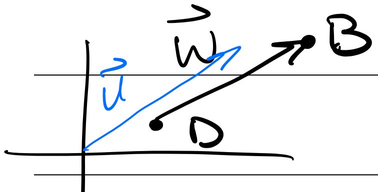
#**5** a. Find the terminal point of the vector that is equivalent to $\mathbf{u} = (1,\ 2)$ and whose initial point is $A(1,\ 1)$.
b. Find the initial point of the vector that is equivalent to $\mathbf{u} = (1,\ 1,\ 3)$ and whose terminal point is $B(-1,\ -1,\ 2)$.

a.



$\vec{w} = (1+1,\ 1+2) = (2,3)$

b. (formal approach) Let $D = (a,b,c)$ be the initial point of $\vec{w}$.



coords of B − coords of D

$\vec{u} = (-1-a,\ -1-b,\ 2-c) = (1,1,3)$

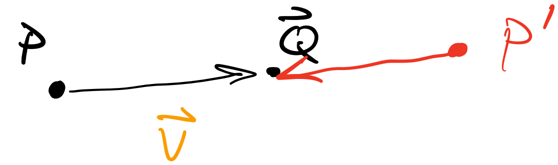
$-1-a=1 \Rightarrow a=-2,\quad -1-b=1 \Rightarrow b=-2$

$2-c=3 \Rightarrow c=-1$ so $D(-2,-2,-1)$

#7 Find the initial point P of a nonzero vector $\mathbf{u} = \overrightarrow{PQ}$ with terminal point $Q(3,\ 0,\ -5)$ and such that

a. $\mathbf{u}$ has the same direction as $\mathbf{v} = (4,\ -2,\ -1)$.

b. $\mathbf{u}$ is oppositely directed to $\mathbf{v} = (4,\ -2,\ -1)$.



a) Let $P(a,b,c)$. Then $\vec{u} = (3-a, -b, -5-c)$

$3-a = 4 \Rightarrow a = -1$, $-b = -2 \Rightarrow b = 2$, $-5-c = -1$

$c = -4 \Rightarrow P(-1, 2, -4)$

b) Let $P(a,b,c)$. Then $(3-a, -b, -5-c) = (-4, 2, 1)$

$3-a = -4 \Rightarrow a = 7$, $b = -2$, $-5-c = 1 \Rightarrow c = -6$

$P'(7, -2, -6)$

Definition: If $\mathbf{v} = (v_1, v_2, \ldots,\ v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots,\ w_n)$ are vectors in R^n, and if k is any scalar, then we define

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots,\ v_n + w_n)$ [this is **vector addition**]
- $k\mathbf{v} = (kv_1, kv_2, \ldots,\ kv_n)$ [this is **scalar multiplication**]
- $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$
- $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots,\ w_n - v_n)$

#12 Let $\mathbf{u} = (1,\ 2,\ -3,\ 5,\ 0)$, $\mathbf{v} = (0,\ 4,\ -1,\ 1,\ 2)$, and $\mathbf{w} = (7,\ 1,\ -4,\ -2,\ 3)$. Find the components of

a. $\mathbf{v} + \mathbf{w}$

b. $3\,(2\mathbf{u} - \mathbf{v})$

c. $(3\mathbf{u} - \mathbf{v}) - (2\mathbf{u} + 4\mathbf{w})$

d. $\frac{1}{2}\,(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v}$

a) $\vec{v}+\vec{w}=(0+7, 4+1, -1-4, 1-2, 2+3)$

$=(7,5,-5,-1,5)$

b) $3(2-0, 4-4, -6+1, 10-1, 0-2)$

$=(6,0,-15,27,-6)$

c) $(3\vec{u}-\vec{v})-(2\vec{u}+4\vec{w})$

$=(-27,-6,14,12,-14)$

$\vec{x}=(x_1, x_2, \dots, x_n)$

The **zero vector**, $\mathbf{0}=(0, 0, \dots, 0)$, has length zero.

$R^2: \vec{0}=(0,0)$, $R^3=(0,0,0)$

Theorem 3.1.1 If **u**, **v**, and **w** are vectors in R^n, and if k and m are scalars, then:

a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})=\mathbf{u}+\mathbf{v}+\mathbf{w}$

c) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

e) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

f) $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

g) $k(m\mathbf{u})=(km)\mathbf{u}$

h) $1\mathbf{u}=\mathbf{u}$

Based on properties of real #s, some of which are $(a,b,c\in\mathbb{R})$

- commutative of addition : $a+b=b+a$
- associative of addition : $(a+b)+c=a+(b+c)$ $=a+b+c$
- dist. : $a(b+c)=ab+ac$

Pf: (f) Let $\vec{u}=(u_1, u_2, \dots, u_n)$, $k, m\in\mathbb{R}$

$(k+m)\vec{u}=((k+m)u_1, (k+m)u_2, \dots, (k+m)u_n)$

by def of scalar mult.

$= (ku_1 + mu_1, ku_2 + mu_2, \dots, ku_n + mu_n)$

by dist prop of real #s

$= (ku_1, ku_2, \dots, ku_n) + (mu_1, mu_2, \dots, mu_n)$

by def. of vector addition

$= k\vec{u} + m\vec{u}$ (def of scalar mult.)

So $(k+m)\vec{u} = k\vec{u} + m\vec{u}$

Theorem 3.1.2 If **v** is a vector in R^n and k is a scalar, then:

a) $0\mathbf{v} = \mathbf{0}$ handwritten: $0\vec{v} = \vec{0}$

b) $k\mathbf{0} = \mathbf{0}$

c) $(-1)\mathbf{v} = -\mathbf{v}$

Definition: if **w** is a vector in R^n, then **w** is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in R^n if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_r\mathbf{v}_r$, where each k_i is a scalar. The scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, we have $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is a scalar multiple of the vector.

#17 Let $\mathbf{u} = (1, -1, 3, 5)$ and $\mathbf{v} = (2, 1, 0, -3)$. Find scalars a and b so that $a\mathbf{u} + b\mathbf{v} = (1, -4, 9, 18)$.

$$a\begin{bmatrix}1\\-1\\3\\5\end{bmatrix} + b\begin{bmatrix}2\\1\\0\\-3\end{bmatrix} = \begin{bmatrix}1\\-4\\9\\18\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2\\-1 & 1\\3 & 0\\5 & -3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\-4\\9\\18\end{bmatrix}$$

$a + 2b = 1$

$-a + b = -4$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 3 & 0 & 9 \\ 5 & -3 & 18 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} a & b & \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$a = 3 \qquad b = -1$$

Check: $3\vec{u} - \vec{v} = (3,-3,9,15) - (2,1,0,-3)$
$= (1,-4,9,18)$ ✓

(This is overkill but lets us practice a matrix approach.)